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MATHEMATICAL THEORY OF LAMINAR COMBUSTION. III. PERTURBATIONS. (U)

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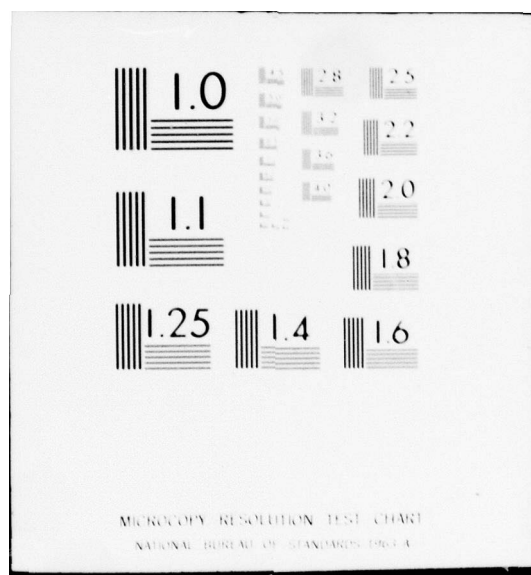
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MATHEMATICAL THEORY OF LAMINAR COMBUSTION. III. PERTURBATIONS

J. D. Buckmaster and G. S. S. Ludford

Technical Summary Report # 1819  
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ABSTRACT

Perturbations of a plane premixed flame propagating steadily along a uniform duct (cf. MRC TSR #1818) are considered. These may be caused by heat loss through the sides of the duct, for which conditions are steady, or by slight variations in the cross section, when a slowly varying flame results. Such unsteadiness may also be self-induced. The results, which concern quenching, stability and velocity changes as well as the behavior of an elementary burner, are remarkably sensitive to whether the Lewis number is greater or less than one.

AMS (MOS) Subject Classification: 76.35

Key Words: Perturbations, Lewis Number, Slowly Varying Flames, Heat Loss, Stability, Non-Uniform Ducts, Elementary Flame Holder

Work Unit Number 3 (Applications of Mathematics)

# SIGNIFICANCE AND EXPLANATION

In any practical situation there are always imperfections to affect a steady state and it is important to determine how they do so. Here we investigate disturbances of the steady progress of a flame along a duct, a problem previously solved in MRC TSR #1818. It is found that small imperfections can cause large deviations of a kind that depends on whether the reactant diffuses more or less rapidly than thermal energy. In particular it is shown how imperfections can be used to stabilize a flame.

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### MATHEMATICAL THEORY OF LAMINAR COMBUSTION. III. PERTURBATIONS

J. D. Buckmaster and G. S. S. Ludford

#### 1. Introduction

The most striking feature of the result (II.22)<sup>\*</sup> is that, for fixed  $D$  and  $J_s$ , changes in  $T_\infty$  produce much larger changes in  $M$  because of the  $\theta$  in the exponential. More precisely, an  $O(\theta^{-1})$  change in flame temperature, due to such a change in  $T_\infty$  for the unbounded flame, produces an  $O(1)$  change in the burning rate. Such a perturbation in flame temperature may also be caused by heat loss through the sides of a uniform duct along which a flame is propagating, so that we may expect a similar change in its speed.

Perturbations may also be produced by slight variations in cross section of the duct, giving rise to the new feature of unsteadiness. Now there will be slow variations in the combustion field, developing on a time scale  $O(\theta)$ . Indeed such slow variations can be self-induced by residual perturbations of the initial conditions (on that time scale) in the absence of boundary perturbations. In all such cases an obvious conjecture is that the flame velocity is not close to the unperturbed value.

It is not so obvious that slowly varying flames behave quite differently according as the Lewis number is smaller or greater than one. Moreover,  $L = 1$  is exceptional; in particular there are no self-induced variations. In order to develop the theory fully, it is necessary to re-do Sec. II.4 for general Lewis number, and that is where we start. The remainder of the chapter is an introduction to slowly varying flames, unearthed by a perturbation theory which is proving increasingly useful in understanding the true role of such idealized solutions as the premixed plane flame.

#### 2. Modifications for $L \neq 1$

The analysis follows that in Sec. II.4<sup>\*</sup> closely. The flame sheet is again located at  $x = x_*$ ; beyond it the temperature is still constant and there is no reactant.

<sup>\*</sup> This notation refers to the equation or section number in Part II, MRC TSR #1818.



Because there is no Shvab-Zeldovich variable, the latter fact must now be deduced from

$$(1) \quad Q(Y, f) = -Q(T, 1) \quad \text{everywhere}.$$

For the same reason both  $Y$  and  $T$  must be carried, and we write

$$(2) \quad Y = Y_S + f^{-1} Y'_S (e^{fx} - 1), \quad T = T_S + T'_S (e^x - 1) \quad \text{for } 0 < x < x_*,$$

where  $Y_S, Y'_S, T_S, T'_S$  are values at  $x = 0$  determined by how the reactant is supplied there. (Fig. II.1 still gives the general shape of these profiles.) Continuity at the flame sheet now shows that

$$(3) \quad x_* = f^{-1} \ln(1 - f Y'_S / Y'_S) = \ln[1 + (T_\infty - T_S) / T'_S].$$

The discussion of reactant supplies in Sec. II.3 showed that only three conditions are provided at  $x = 0$ , so that we may expect  $Y'_S$  and  $T_\infty$  (say) to be related to  $Y_S, T_S, T'_S$ . The result (3) provides one such relation and the second, namely

$$(4) \quad T_\infty = Y_S + T_S - (f^{-1} Y'_S + T'_S) = J_S + E_S$$

comes from a heat balance between the stations  $x = 0$  and  $\infty$  [formally by integrating equation (1)]. The second enables the first to be re-written

$$(5) \quad f^{-1} Y'_S (1 - f Y'_S / Y'_S)^{1-1/f} = -T'_S,$$

so that now the reductions (II.6) and  $Y'_S = -T'_S$  for  $f = 1$  are clear.

To ensure  $0 < x_* < \infty$  we must have  $Y'_S < 0$  and hence  $T'_S > 0$ : whatever supplies the reactant must still be a (conductive) heat sink. Since the  $T$ -profile is monotonic, it follows that the temperature of the flame is always higher than that of the supply, something that is not obvious from the relations (4) and (5) when  $f \neq 1$ .

$Y_S + T_S$  is the adiabatic flame temperature, so-called because it is the final temperature when the reactant is allowed to decompose under adiabatic conditions. However, the system receives heat  $-f^{-1} Y'_S$  by diffusion and loses heat  $T'_S$  by conduction, where the relation (5) shows

$$(6) \quad -f^{-1} Y'_S \begin{matrix} > \\ < \end{matrix} T'_S \quad \text{according as } f \begin{matrix} < \\ > \end{matrix} 1.$$

Thus the actual flame temperature (4) is higher or lower than the adiabatic.

Here the expansions (II.17) lead to the structure equations

$$(7) \quad \mathcal{L}^{-1} d^2 y / d\xi^2 = -d^2 t / d\xi^2 = \tilde{\Lambda} y e^t,$$

which matching requires to be solved under the boundary conditions

$$(8) \quad y = -\mathcal{L} J_S \xi + o(1), \quad t = J_S \xi + o(1) \quad \text{as } \xi \rightarrow -\infty; \quad y, t = o(1) \quad \text{as } \xi \rightarrow +\infty,$$

where

$$(9) \quad J_S = Y_S - \mathcal{L}^{-1} Y'_S.$$

now. We immediately conclude that there is a local Shvab-Zeldovich variable and that it stays constant through the layer, more precisely

$$(10) \quad t + \mathcal{L}^{-1} y = 0.$$

The first integral (II.20) is therefore replaced by

$$(11) \quad (dt/d\xi)^2 = 2\mathcal{L}\tilde{\Lambda}[1 + (t-1)e^t]$$

and the result (II.22) by

$$(12) \quad \tilde{\Lambda} \equiv DM^{-2} = J_S^2 \theta^2 \exp(\theta/T_\infty) / 2\mathcal{L} T_\infty^4,$$

but the structure (II.23) still holds. The formula (12), as well as its correction of relative order  $\theta^{-1}$ , is due to Bush & Fendell (1970).

The modifications to Secs. II.5 and 6 which have to be made when  $\mathcal{L} \neq 1$  are quite similar. Of more importance for our purposes is the form which the results take for the unbounded flame. By transferring the origin to the flame sheet and then letting  $T'_S \rightarrow 0$  we find

$$(13) \quad Y = Y_\infty (1 - e^{\mathcal{L}x}), \quad T = T_\infty + Y_\infty e^x \quad \text{for } x < 0,$$

$$(14) \quad Y = 0, \quad T = T_\infty = Y_\infty + T_\infty \quad \text{for } x > 0$$

and the eigenvalue (12) with  $J_S = Y_\infty$  again. Note that the adiabatic flame temperature obtains here because there are no fluxes into or out of the system.

Suppose now that  $T_\infty$  is increased by  $\delta t_\infty$ . Then the flame temperature changes to

$$(15) \quad T_* = T_\infty + \delta t_\infty$$

and the formula (12) shows that



$$(16) \quad M = M_0 e^{t_\infty/2} \quad \text{where} \quad M_0 = \sqrt{2LD} T_\infty^2 \exp(-\theta/2T_\infty) / J_s \theta$$

is the burning rate for the original flame temperature  $T_\infty$ . This result has universal validity, i.e. it holds whatever the nature of the perturbation, steady or not. The reason is that in determining the eigenvalue, the perturbation only intrudes through the matching of  $t$  at  $\xi = +\infty$  (which leads to the exponential factor).

### 3. Theory of Perturbations

The equations that we shall discuss are

$$(17) \quad \partial \rho / \partial t + \partial(\rho v) / \partial x = \delta f,$$

$$(18) \quad \rho \partial Y / \partial t + \rho v \partial Y / \partial x - L^{-1} \partial^2 Y / \partial x^2 = -\Lambda Y \exp(-\theta/T) + \delta g,$$

$$(19) \quad \rho \partial T / \partial t + \rho v \partial T / \partial x - \partial^2 T / \partial x^2 = \Lambda Y \exp(-\theta/T) + \delta h.$$

Here  $f, g, h$  should be regarded, for the moment, as arbitrary functionals; note that the Lewis number has been reinstated. For propagation into a quiescent gas,

$$(20) \quad Y, T, v \rightarrow Y_\infty, T_\infty, 0 \quad \text{as} \quad x \rightarrow -\infty.$$

The units by which these equations have been made dimensionless are based on a representative mass flux, which will be taken as that through the flame sheet in the absence of perturbations, namely the  $M_0$  of equations (16); so that  $\Lambda = DM_0^{-2}$  in the above equations.

To avoid an unnecessary factor we have changed the unit of density from  $\rho_c$  to  $\rho_\infty \rho_c = \rho_c / T_\infty$  (with corresponding changes in other units). The incoming velocity is then 1 and  $T_\infty$  appears in the equation of state, i.e.

$$(21) \quad \rho = T_\infty / T.$$

For finite  $\theta$ , it is possible to define a flame speed only when the combustion is steady: when the structure changes with time in all frames of reference, the location of the flame (and hence its speed) is not a precise concept. In the limit  $\theta \rightarrow \infty$ , however, it becomes precise since the reaction is confined to a sheet. With  $x_*(t)$  denoting the position of the sheet, we write

$$(22) \quad s = x_*(t) - x$$

for the distance ahead to obtain

$$(23) \quad \partial \rho / \partial t - \partial [\rho (V+v)] / \partial s = \delta f ,$$

$$(24) \quad \rho \partial Y / \partial t - \rho (V+v) \partial Y / \partial s - L^{-1} \partial^2 Y / \partial s^2 = -\Lambda Y \exp(-\theta/T) + \delta g ,$$

$$(25) \quad \rho \partial T / \partial t - \rho (V+v) \partial T / \partial s - \partial^2 T / \partial s^2 = \Lambda Y \exp(-\theta/T) + \delta h$$

where  $V = \dot{x}_*(t)$  is the flame speed. The unperturbed flame corresponds to  $V = 1$  and we are concerned with flames for which  $V$  stays  $O(1)$ . We may expect that to be the case for  $f, g, h$  all  $O(1)$  and, when conditions are unsteady,  $\partial/\partial t = O(\delta)$  i.e. slowly varying flames. Then the result (16a), which now reads

$$(26) \quad V = e^{t_\infty/2}$$

because  $\rho$  is not changed to  $O(1)$ , shows that our task is to evaluate the increase  $\delta t_\infty$  in flame temperature due to the perturbations. Before doing that, we note for future reference that

$$(27) \quad Y \equiv 0 \text{ behind the flame sheet .}$$

If it were not, the reaction term in equation (70) would be  $O(\theta^2)$ , since  $T = T_\infty$  to leading order, and hence unbalanced. Otherwise stated,  $Y$  relaxes to zero in a time  $O(\theta^{-2})$ .

The first calculation of  $t_\infty$  was for distributed heat loss, which was considered independently by Buckmaster (1976) and by Joulin & Clavin (1976) using straightforward matching of 3 terms in the reaction zone with 2 terms on either side, i.e. one more term in each expansion than is required without loss. Other perturbations were later treated by Buckmaster (1977), who by then realized that  $t_\infty$  (for whose determination the extra terms are introduced) can be calculated directly from the overall change in enthalpy of the mixture up to and including the flame sheet.

The direct calculation is as follows. If equations (24) and (25) are added to eliminate the reaction terms and then integrated from  $s = 0^-$  to  $\infty$ , we obtain

$$\int_{0^-}^{\infty} \rho (V+v) \frac{\partial (Y+T)}{\partial s} ds = \left. \frac{\partial T}{\partial s} \right|_{0^-} + \int_{0^-}^{\infty} \left[ \rho \frac{\partial (Y+T)}{\partial t} - \delta g - \delta h \right] ds ,$$

correct to  $O(\delta)$ , since  $Y$  vanishes behind the flame sheet. Integrating by parts on the left-hand side and using the continuity equation (23) now yields

$$(T_{\infty} - T_*)V = \left. \frac{\partial T}{\partial s} \right|_{0-} + \int_{0-}^{\infty} \left[ (Y - T_*) \frac{\partial \rho}{\partial t} + \rho \frac{\partial Y}{\partial t} - (Y + T - T_*) \delta f - \delta g - \delta h \right] ds$$

in view of the equation of state (73). Here  $T_* = T(0-, t)$  may properly be called the flame temperature, while  $T_{\infty}$  is the adiabatic flame temperature  $Y_{\infty} + T_{\infty}$ . Thus

$$(28) \quad t_{\infty} V = - \left. \frac{1}{\delta} \frac{\partial T}{\partial s} \right|_{0-} + \int_0^{\infty} \left[ (T_{\infty} - Y) \frac{\partial \rho}{\partial \tau} - \rho \frac{\partial Y}{\partial \tau} + (Y + T - T_{\infty}) f + g + h \right] ds,$$

where

$$(29) \quad \tau = \delta t$$

is a long time (first introduced by Sivashinsky, 1974) and all terms are to be evaluated to leading order only.

The whole problem is reduced to calculating the right-hand side of equation (28) in terms of  $V$  for the particular perturbation of interest. It then becomes an equation for the flame speed (the goal of our analysis) since the universal formula (26) shows the left-hand side to be  $V \ln V^2$ .

#### 4. Steady Heat Loss

In practice the mixture must be confined laterally and there is then the possibility of losing heat to the sides. For example, heat conduction through the wall of the tube introduced in Sec. II.2 will result in a heat loss proportional to  $T - T_{\infty}$ , if we assume that the temperature outside the tube at any point is that of the mixture at  $x = -\infty$ . Similarly, thermal radiation (as distinct from chemiluminescence) is present, though usually negligible, in combustion processes, resulting in loss proportional to  $T^4 - T_{\infty}^4$ . The importance of considering such effects comes from the observed inability of the mixture to ignite if the tube is too thin, a phenomenon about which our analysis so far has nothing to say. Then conductive loss, though not radiative, is significant.

Heat losses are normally minimized as much as possible, so that the proportionality factors are small. Our analysis will therefore be based on  $O(\theta^{-1})$  factors, but the dependence of the loss on temperature will be quite general; in particular, both conductive

and radiative losses are covered. A term  $-\delta M^{-2} \phi(T)$  is added to the right-hand side of the temperature equation (II.4), where  $\phi$  vanishes at  $T_{-\infty}$  but is otherwise an arbitrary positive function, and  $\delta$  has the definition (II.16). (The factor  $M^{-2}$  ensures that  $\phi$  is independent of the mass-flux and density units.) In particular,

$$(30) \quad \phi = \begin{cases} k(T - T_{-\infty}) & \text{for conductive losses,} \\ k(T^4 - T_{-\infty}^4) & \text{for radiative losses} \end{cases}$$

where  $k$  is a given constant.

We are therefore dealing with the special case of equations (23-25) in which

$$(31) \quad \partial/\partial \tau = f = g = 0, \quad h = -M_0^{-2} \phi(T).$$

The continuity equation integrates to

$$(32) \quad \rho(V+v) = V,$$

with  $V$  constant, so that to leading order we are left with the unperturbed steady equations. Use can therefore be made of the result (13) to write

$$(33) \quad Y = Y_{-\infty}(1 - e^{-Lv_s}), \quad T = T_{-\infty} + Y_{-\infty} e^{-Vs} \quad \text{for } s > 0,$$

but (14) must be taken one term further. From the perturbed temperature equation we easily find

$$(34) \quad T = T_{\infty} + \delta[t_{\infty} + V^{-1} \phi(T_{\infty})s] \quad \text{for } s < 0,$$

a result that is needed to calculate  $\partial T/\partial s$  at  $0^-$ . [An exponentially growing term has been discarded as unmatchable with the decay to ambient temperature which occurs on a scale  $O(\theta)$ .]

The right-hand side of (28) may now be calculated, giving

$$(35) \quad t_{\infty} = -\phi V^{-2} \quad \text{where} \quad M_0^2 \phi = \phi(T_{\infty}) + \int_0^{\infty} \phi(T_{-\infty} + Y_{-\infty} e^{-\sigma}) d\sigma,$$

so that

$$(36) \quad V^2 \ln V^2 + \phi = 0$$

is the equation for the flame speed. The result is graphed in Fig. 1; such C-shaped response curves are common in combustion theory. The function  $\phi$  consists of two terms:



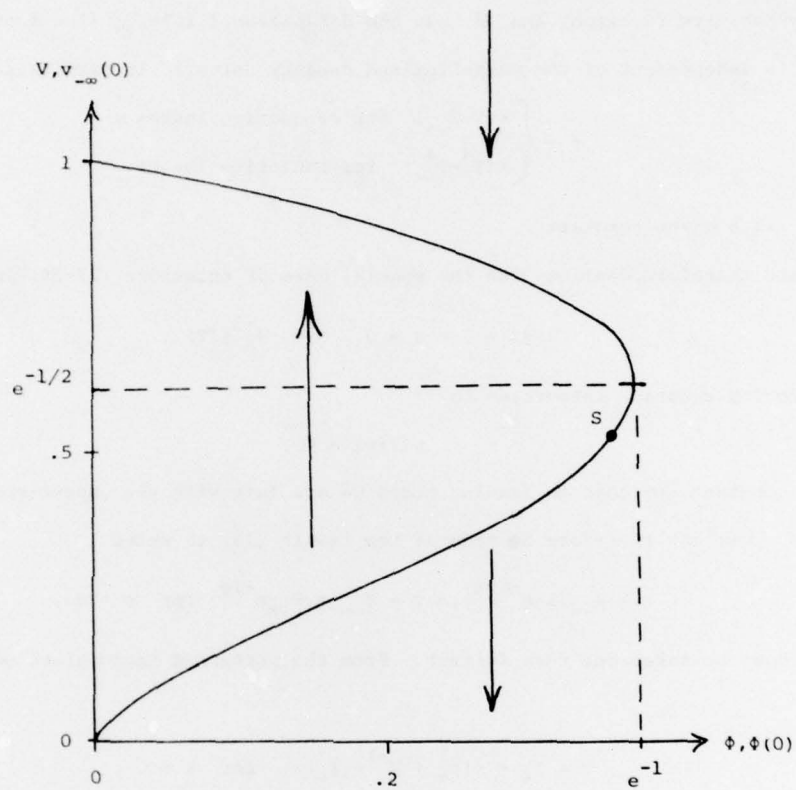


Fig. 1: Flame velocity  $V$  versus heat-loss parameter  $\phi$ .

The notation  $v_{-\infty}(0)$  and  $\phi(0)$  refers to Sec. 7

where  $S$ , the point at which  $\phi(0) = 4v_{-\infty}^2(0)/3$ ,

plays a role.



the integral represents the total heat loss up to the flame while the other accounts for conduction into the burnt mixture behind.

The top branch shows that heat loss decreases the flame velocity, though if  $\phi$  exceeds the critical value  $e^{-1}$  there is no corresponding  $V$ , i.e. no flame. The term extinction is therefore appropriate for the point at which the curve turns back, the burning rate then being  $e^{-1/2}$  ( $= 0.61$ ) times the adiabatic value for the same pressure. (Note that this result is independent of all parameters.) The bottom branch is generally believed to be unstable (Emmons, 1971) and hence of no interest, but we shall postpone a discussion of that point until the next section.

It has been known for over 150 years that a flame will not propagate through a wire gauze, the principle underlying the miner's safety lamp invented by Davy. This is suggested by the present results: a cold wire gauze of sufficiently small mesh is an effective heat sink and corresponds to values of  $\phi$  greater than critical.

Extinction in small tubes can now be explained. Heat loss by conduction through the wall is proportional to surface area and hence the radius, while the volume of the mixture is proportional to its square. The heat loss per unit volume therefore varies inversely as the radius, and hence so does the parameter  $\phi$ . There is therefore a definite tube radius, independent of pressure, below which steady combustion will not occur.

##### 5. Slowly Varying Flames

Flames can be unsteady for a variety of reasons. Evolution from ignition, passage down a tube of varying cross section and instability are all examples of unsteady combustion. Here we are concerned with the one-dimensional problem of a flame evolving from initial data; in general, such an evolution can be expected to take place on an  $O(1)$  time scale. Indeed there have been numerical integrations (with  $f, g, h$  all zero) that show the emergence of what appears to be a steady wave from a mass of hot quiescent gas on such a time scale (Zeldovich & Barenblatt, 1959).

Our techniques cannot describe such phenomena in general since the right side (28) will not be small. However, from the initial development there emerges a structure,

characterized by the single variable  $V$ , that undergoes a slower evolution on the time scale  $\theta$ , and during this phase of the unsteady process the time derivatives are perturbations that can be incorporated into the analysis as easily as heat loss, for example.

To leading order the continuity equation still has the integral (32), where  $V$  is now a function of  $\tau$ , so that the profiles (33) remain valid ahead of the flame sheet. To them we must add

$$(37) \quad \rho = T_{-\infty} / (T_{-\infty} + Y_{-\infty} e^{-Vs})$$

but an expression for  $v$  will not be needed. As before, the remaining question is to calculate  $\partial T / \partial s$  at  $0^-$ , which comes from the temperature perturbation behind the flame sheet.

To begin with, consider perturbations due solely to the unsteadiness. Then  $f = g = h = 0$  and hence

$$(38) \quad T = T_{\infty} + \delta t_{\infty} \quad \text{for } s < 0,$$

so that

$$(39) \quad t_{\infty} = -bV^{-3} \dot{V} \quad \text{where } b = Y_{-\infty} T_{-\infty} \int_0^1 \frac{\sigma^{f-1} - 1}{T_{-\infty} + Y_{-\infty} \sigma} d\sigma$$

and the dot denotes rate of change with respect to the time  $\tau$ . The equation for  $V(\tau)$  is therefore

$$(40) \quad b\dot{V} + V^3 \ln V^2 = 0.$$

Note that the parameter  $b$  changes from positive to negative as the Lewis number increases through one, giving the first indication that  $f = 1$  is not typical.

We are only free to specify the initial value of  $V$  and then the present approximation to the combustion field is uniquely determined. As mentioned previously, the present approach does not solve the general initial-value problem since the initial evolution takes place on the  $O(1)$  time scale. But from this initial development emerges a structure, characterized by the single variable  $V$ , that changes on the  $O(\theta)$  time scale.

When  $f > 1$ ,  $b$  is negative, and when  $f < 1$ ,  $b$  is positive. The closer  $f$  is to 1 the more rapidly  $V$  changes, so that when  $f - 1 = O(1/\theta)$  or smaller there are no  $O(1)$

effects on the slow-time scale and the result (36) implies

$$(41) \quad V = 0 \text{ or } 1.$$

We infer that the initial evolution ends at one of these values, though we may expect the ultimate value to be reached more slowly in  $t$  as the Lewis number gets further from 1 on the scale of  $1/\theta$ . It is tempting to take unit Lewis number in combustion problems because of the simplifications stemming from global Shvab-Zeldovich variables (Sec. I.7); but in view of the present atypical behavior such temptations should certainly be resisted for unsteady problems.

The values (41) are also possible steady states as  $\tau \rightarrow \infty$ . However, the first corresponds to  $t_\infty = \infty$ , when our analysis breaks down but indicates that the flame temperature differs from its adiabatic value by an  $O(1)$  amount. A discussion of the non-uniformity, in the context of vanishingly small heat loss, has been given by Buckmaster (1976). The second is unstable when  $b$  is negative, i.e.  $L > 1$ . For, according to (40), when  $V$  is less than 1 it will be driven towards 0 and when  $V$  is greater than 1 it will be driven towards  $\infty$ .

It might be thought, therefore, that steady flames of Lewis number greater than 1 are of no practical interest; but laboratory experience shows otherwise. Moreover, if flames with  $L > 1$  are to be discarded on these grounds then so must flames with  $L < 1$ , since they are also unstable when three-dimensional disturbances are permitted (Ch. VI). The fact that flames are stabilized by burners suggests that velocity and thermal gradients play an important role. Indeed, it is not difficult to devise a one-dimensional model (Sec. 10) that stabilizes flames in this way. However, it is unlikely that such mechanisms will explain stable flames propagating down uniform tubes.

To be sure, Lewis numbers are usually not far from 1 so that, in view of the irrelevance of the slow-time scale when  $L - 1 = O(1/\theta)$ , it is conceivable that predictions of a slow-time analysis are often not applicable. Alternatively, stable flames may be a phenomenon of finite activation energy about which asymptotic theory provides faulty information. Finally, stable flames may be the result of chemical kinetics more

complicated than that adopted here. Whatever the explanation, activation-energy asymptotics for  $f \neq 1$  can provide useful qualitative insights even though the applicability of the model cannot be justified at the present time.

To determine the effect of heat losses we still let  $f = g = 0$  but now take  $h = -M_0^{-2} \phi(T)$ . The effects of unsteadiness and heat loss are then seen to be additive, i.e.

$$(42) \quad t_\infty = -bV^{-2} \dot{V} - \phi V^{-2},$$

so that we end with

$$(43) \quad b\dot{V} + V^3 \ln V^2 + \phi V = 0$$

as the equation governing  $V$ . To the possible steady state (87), plotted in Fig. 1, must now be added

$$V = 0 \text{ for all } \phi.$$

However, the latter corresponds to  $t_\infty = \infty$ , a non-uniformity which has yet to be treated.

The question of which branch of the multivalued response in Fig. 1 will be observed in practice can be answered in the context of our present model. The arrows in Fig. 1 give the direction in which  $V$  changes according to the differential equation (43) when  $b$  is positive (i.e.  $f < 1$ ). The lower branch of (36) is therefore predicted to be unstable, a result in accord with general belief. However, Spalding & Yumlu (1959) claim to have observed the slow branch which, in the absence of other effects, can only be reconciled with our analysis if  $b$  is negative (i.e.  $f > 1$ ) when the arrows are reversed. [The fact that they used a special stabilizing apparatus suggests that there were other effects, cf. Sec. 7.]

## 6. Non-Uniform Ducts

We consider ducts whose cross-sectional area  $A$  varies over distances of order  $\theta$ , i.e.

$$(44) \quad A = A(\delta x) = A\left(\int_0^T V d\tau - \delta s\right).$$

In the usual one-dimensional treatment of such slowly varying ducts, the divergences in the basic equations (I.53,54,56) now give rise to extra terms



$$-\left(\frac{1}{A} \frac{\partial A}{\partial s}\right) \rho v, \quad -\mathcal{L}^{-1} \left( \frac{1}{A} \frac{\partial A}{\partial s} \right) \frac{\partial Y}{\partial s}, \quad -\left(\frac{1}{A} \frac{\partial A}{\partial s}\right) \frac{\partial T}{\partial s},$$

which correspond to

$$(45) \quad f = F(X) \rho v, \quad g = \mathcal{L}^{-1} F(X) \partial Y / \partial s, \quad h = F(X) \partial T / \partial s,$$

where

$$(46) \quad F(X) = \mathcal{Q}'(X) / \mathcal{Q}(X) \quad \text{with} \quad X = \int_0^T V d\tau.$$

We have neglected  $\delta s$  in the argument of  $\mathcal{Q}$  for  $s = 0(1)$ , i.e.  $F$  is the logarithmic derivative of the area at the location  $X$  of the flame sheet.

It is easily seen that  $\partial T / \partial s = 0$  at  $0^-$ , so that the extra terms

$$(47) \quad F(X) \int_0^\infty [\mathcal{L}^{-1} \partial Y / \partial s + \partial T / \partial s + (Y + T - T_\infty)(1 - \rho)V] ds = -bF(X)$$

in  $t_\infty V$  come from the integral. They lead to

$$(48) \quad b\dot{V} + V^3 \ln V^2 - bF(X)V^2 = 0$$

as the replacement for the governing equation (91).

An immediate consequence is that a flame sheet propagating steadily ( $V = 1$ ) down a duct of constant cross section will in general accelerate on entering a diverging section ( $\mathcal{Q}' > 0$ ); whereas if it enters a converging section ( $\mathcal{Q}' < 0$ ) it will slow down (Sivashinsky, 1974). The only exception is  $b = 0$  ( $\mathcal{L} = 1$ ), for then  $V = 1$  irrespective of area changes. Again,  $\mathcal{L} = 1$  is exceptional.

The  $F(X)$  makes the differential equation (48) very difficult to discuss in general. However, if the area changes very slowly indeed, more precisely  $\mathcal{Q}(X) = \tilde{\mathcal{Q}}(\epsilon X)$  where  $0 < \epsilon \ll 1$ , there is an approximate solution

$$(49) \quad V = 1 + \epsilon b \tilde{F}(\epsilon X), \quad \tilde{F} = \tilde{\mathcal{Q}}' / \tilde{\mathcal{Q}}$$

in which  $V$  differs just a little from 1. It represents a balance between the effect of area change and that of enthalpy deficiency at the flame, temporal variations playing no role.

There have been attempts to discuss the effect of curvature on unsteady flames by supposing that the difference between the speed of a slightly curved flame and a plane



flame is proportional to the curvature, with the constant of proportionality depending only on the mixture properties and not on the particular unsteady process. Markstein (1964, p. 22) has discussed flame stability in this way. The result (49) provides a rational justification for such an assumption, when it is realized that curvature manifests itself through fractional changes in area. In particular, it shows that the constant of proportionality changes sign as  $L$  passes through 1.

### 7. An Elementary Flame Holder

An extension of the present discussion provides some insight into the nature of flame holders or burners. The basic function of a burner is to maintain a flame in a stable rest position: the flame is brought to rest by applying a counterflow; stability is ensured by appropriate velocity and thermal gradients. Velocity gradients can be produced by expansion of the mixture as it leaves a tube or by flow to the rear of a bluff body, while thermal gradients arise quite naturally by conduction to the burner, to mention some common examples. These can be modelled by a one-dimensional formulation of propagation along a non-uniform duct with a counterflow, incorporating heat losses which depend on location as well as temperature. For simplicity we shall only consider conductive losses through the side walls for which the linear law (30a) holds with  $k$  a slowly varying function of  $x$ , just as  $A$  is. The perturbation terms are now

$$(50) \quad f = F(X)\rho v, \quad g = L^{-1}F(X)\partial Y/\partial s, \quad h = F(X)\partial T/\partial s - M_0^{-2}k(X)(T-T_{-\infty}).$$

The counterflow changes the velocity condition ahead of the flame to

$$(51) \quad v \rightarrow v_{-\infty} \quad \text{as} \quad x \rightarrow -\infty$$

so that continuity now requires

$$(52) \quad \rho(v+V) = v_{-\infty} + V = \mathcal{V} \quad (\text{say})$$

to leading order. At various places in the preceding analysis we must now replace  $V$  by  $\mathcal{V}$ . As a consequence the expansions (27), (33), (34), (37) and the enthalpy result (42) with the extra term (47) still apply if  $V$  is everywhere replaced by  $\mathcal{V}$  except in  $(1-\rho)V$ , which must be replaced by  $\mathcal{V} - \rho V$ . Noting that

$$(53) \quad \phi = k(X) (T - T_{-\infty})$$

then gives

$$(54) \quad t_{\infty} \eta = -b\eta^{-2} \eta' - \phi(X) \eta^{-1} + bF(X) V \eta^{-1} \quad \text{where} \quad M_0^2 \phi(X) = 2Y_{-\infty} k(X),$$

so that the governing equation becomes

$$(55) \quad b\eta' + \eta^3 \ln \eta^2 + \phi(X) \eta - bF(X) V \eta = 0.$$

Without counterflow, i.e.  $v_{-\infty} = 0$  and  $\eta = V$ , we reach a combination of the two equations (43) and (48) as expected.

In the absence of heat losses ( $\phi \equiv 0$ ) there is a stationary solution

$$V = 0 \quad \text{provided} \quad \eta = 1 \quad \text{i.e.} \quad v_{-\infty} = 1.$$

The flame speed relative to the fresh mixture is the same as that for a uniform duct. That is, the area changes do not affect the flame speed to leading order, in striking contrast to the unsteady problem discussed earlier where there was no counterflow. Of course the difference is that the flame sheet itself does not encounter area changes here. The corresponding condition is

$$(56) \quad v_{-\infty}^2 \ln v_{-\infty}^2 + \phi(0) = 0$$

when heat losses are present. Not surprisingly, this is just the relation (36) with  $V$  replaced by  $v_{-\infty}$  and with  $\phi$  given its value at the origin (i.e. any finite  $x$ ); so that Fig. 1 applies. In either case, with or without heat loss, the location of the flame is indeterminate.

When the flame is moving,  $v_{-\infty}$  becomes a function of  $X$ . For it is the mass flux of the counterflow which is held fixed, whereas the area  $Q(X)$  at the flame varies with its position. That is the area which is effectively constant in the integration (52) of the equation of continuity on the  $x$ -scale, so that

$$(57) \quad v_{-\infty} \propto Q^{-1}(X).$$

Accordingly, we should write  $v_{-\infty}(0)$  in the condition (56).

To investigate the stability of a stationary flame, time dependent solutions of equation (55) are sought in which  $V$  is small (though still large compared to  $\delta$ ), so

that  $\mathcal{V}$  differs just a little from  $v_{-\infty}$ . The linear equation governing small changes from the rest position is

$$(58) \quad \ddot{X} + [b^{-1}D - 2F(0)]v_{-\infty}(0)\dot{X} + [\Phi'(0) - DF(0)v_{-\infty}(0)]b^{-1}v_{-\infty}(0)X = 0,$$

where

$$(59) \quad D = 2v_{-\infty}(0)[1 + 2 \ln v_{-\infty}(0)] = -d\Phi(0)/dv_{-\infty}(0);$$

and the necessary and sufficient condition for stability is that the coefficients of both  $\dot{X}$  and  $X$  are positive. The stability criteria are therefore

$$(60) \quad 2F(0) < b^{-1}D \quad \text{and} \quad \Phi'(0) \geq DF(0)v_{-\infty}(0) \quad \text{according as} \quad b \gtrless 0.$$

To see how these conditions can be met, consider a duct whose cross section varies in size but not in shape. The argument used for a circular tube at the end of Sec. 4 then shows that the heat-loss coefficient varies inversely as a linear dimension of the cross section, i.e.

$$(61) \quad k \propto \mathcal{A}^{-1/2}(X).$$

It follows that  $\Phi'(0) = -\frac{1}{2}\Phi(0)F(0)$ , so that the second criterion (60) may be written

$$(62) \quad \Phi(0)F(0) \gtrless \frac{4}{3}v_{-\infty}^2(0)F(0) \quad \text{according as} \quad b \gtrless 0,$$

since  $Dv_{-\infty}(0) = 2[v_{-\infty}^2(0) - \Phi(0)]$ .

To interpret these conditions, note that the derivative  $D$  is positive/negative on the upper/lower branches in Fig. 1. When the Lewis number is less than 1, i.e.  $b > 0$ , a flame is stable only if it corresponds to a point on the curve above  $S$  and the duct converges; indeed on the lower branch the convergence must be sufficiently rapid, i.e.

$$(63) \quad F(0) < D/2b.$$

When the Lewis number exceeds 1, i.e.  $b < 0$ , there is stability only on the lower branch and then the duct must converge or, for points above  $S$ , diverge sufficiently slowly:

$$(64) \quad F(0) < D/2b.$$

A well-insulated duct must therefore converge and then the flame with  $v_{-\infty}$  small is stable for  $\mathcal{L} > 1$  while that with  $v_{-\infty}$  close to 1 is stable for  $\mathcal{L} < 1$ . The results for a uniform duct in Sec. 5 are a limiting form of these conclusions.

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